Int. J. Solids Structures Vol. 27, No. 11, pp. 1401–1417, 1991
Printed in Great Britain.
0020-7683/91 $3.00 + .00

ASYMPTOTIC VARIATIONAL METHODS IN LARGE DEFLECTION SMALL STRAIN PLATE THEORY

LOWELL J. BERG
IBM Application Business Systems, Hwy 52 and 37th Street, Rochester, MN 55901, U.S.A.

(*Received* 27 *January* 1990; *in revised form* 18 *May* 1990)

Abstract—Several small strain, large deformation plate theories are developed via an asymptotic expansion of the stresses, strains and displacements in a Hu–Washizu variational principle. Both the governing equations and the natural boundary conditions are obtained. The governing two-dimensional plate equations can be obtained from the nonlinear equations governing an elastic three-dimensional continuum. The boundary conditions cannot be obtained except through the use of variational methods. Of special interest are Kirchhoff-type moment boundary conditions, and in-plane boundary conditions similar to those obtained in thin shell theory.

INTRODUCTION

Large deformation small strain plate theories can be derived from the general kinematics and equilibrium equations of nonlinear elasticity. Perturbation methods (Simmonds and Mann, 1986; van Dyke, 1975) can be used to simplify the general equations by separating the dominant terms from those of lesser importance. For a thin plate, the three-dimensional nonlinear elasticity equations can be integrated through the plate's thickness to obtain sets of two-dimensional equations as in Berg and Johnson (1989) and Berg (1988). While the technique of simplifying known general three-dimensional equations via perturbation methods and asymptotic integration is very powerful, it does not lead to a systematic method of obtaining boundary conditions associated with the subsequent two-dimensional equations.

The most convenient way of deriving boundary conditions associated with two-dimensional plate equations is through the use of variational methods. Of particular interest in this investigation are the boundary conditions associated with an inextensible large deformation plate theory.

Inextensible deformations of shallow elastic shells have been studied by Reissner (1961) using a variational formulation. His work is restricted to finite deformations and shallow shells. A second paper by Reissner (1962) also uses a variational formulation to study more general shells, including inextensible shells, but is restricted to small deformations. Ashwell (1963) presents the equilibrium equations of a fictitious edge beam as boundary layer equations for large deformation of very thin plates. Ashwell assumes there are three force and three couple components in the edge beam. By various scaling arguments, he shows that two moment components are negligible, but still retains four unknowns. It is not clear from Ashwell's development that the number of boundary layer equations can be further reduced. Asymptotic expansions are used in the work of Coutris and Monavon (1988), but transverse displacements are restricted to be of the order of the plate's thickness. In-plane displacements are further restricted to be much smaller than the plate's thickness.

This investigation details the derivation of governing equations and boundary conditions appropriate for the lowest order thin plate theory described in Berg and Johnson (1989) and Berg (1988). Conditions similar to the Kirchhoff boundary conditions of ordinary plate theory are obtained. As in the Kirchhoff transverse shear boundary condition, a combination of moment derivatives, transverse shears, in-plane normal stress resultants and applied tractions are required to vanish on a free surface. A large deformation extension of the von Karman plate equations is also derived. Three examples demonstrating the type of problem the different theories govern are also presented.

VARIATIONAL FORMULATION

Consider the functional associated with a Hu–Washizu three-field variational principle for nonlinear elasticity

$$\Pi(E, S, \mathbf{u}; \mathbf{t}) = \int_{V_0} W(E) + \tfrac{1}{2}S_{ij}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} - 2E_{ij}\right) - f_i u_i \, \mathrm{d}V$$
$$- \int_{\partial V_{0u}} (u_i - \hat{u}_i)t_i \, \mathrm{d}A - \int_{\partial V_{0t}} u_i \hat{t}_i \, \mathrm{d}A - \int_{\partial V_{0p}} u_i \hat{p}_i \, \mathrm{d}A \quad (1)$$

where ordinary Cartesian tensor notation is used. E is the Green strain tensor, S is the Piola–Kirchhoff stress tensor of the second kind, and $\mathbf{u}$ is the displacement vector. $W(E)$ is the strain energy function and a circumflex indicates a prescribed field either throughout the plate's interior, V_0, or on its boundary, ∂V_0. The body force per unit undeformed volume is $\mathbf{f}$. Assume the reference configuration for the plate is a rectangular parallelepiped,

$$0 \leqslant X_1 \leqslant l_x, \quad 0 \leqslant X_2 \leqslant l_y, \quad -\frac{h}{2} \leqslant X_3 \leqslant \frac{h}{2}.$$

The boundary of V_0 is composed of three disjoint sets: the faces ($X_3 = \pm h/2$) where tractions $\hat{p}_i$ are prescribed, the edges where tractions $\hat{t}_i$ are prescribed, and the edges where displacements $\hat{u}_i$ are prescribed.

Define the scaling parameter ε as

$$\varepsilon = \frac{h}{L} \ll 1, \quad l_x = O(L), \quad l_y = O(L), \quad h = O(\varepsilon L)$$

then h, the plate's thickness, is much smaller than a typical lateral dimension. Greek indices will take on the values 1 and 2, while Latin indices will take on the values 1, 2 and 3. Summation will be assumed over repeated Latin or Greek indices. The material coordinates are rescaled as follows:

$$X_\alpha = L\xi_\alpha, \quad X_3 = \varepsilon L \xi_3, \quad \xi_i = O(1)$$
$$\frac{\partial}{\partial X_\alpha} = \frac{1}{L}\frac{\partial}{\partial \xi_\alpha} = \frac{1}{L}(\ )_{,\alpha}$$
$$\frac{\partial}{\partial X_3} = \frac{1}{\varepsilon L}\frac{\partial}{\partial \xi_3} = \frac{1}{\varepsilon L}(\ )_{,3}.$$

Next, assume the displacements u_i and $\hat{u}_i$ have the following asymptotic expansions:

$$u_i(\xi_j, \varepsilon) = L(u_i^0(\xi_j) + \varepsilon u_i^1(\xi_j) + \varepsilon^2 u_i^2(\xi_j) + \cdots)$$
$$\hat{u}_i = L(\hat{u}_i^0 + \varepsilon \hat{u}_i^1 + \varepsilon^2 \hat{u}_i^2 + \cdots)$$
$$u_i^m = O(1), \quad \hat{u}_i^m = O(1), \quad m = 0, 1, 2, \ldots \quad (2)$$

The superscript on ε is an exponent, but the superscript on the displacement functions merely identifies the function with a particular ordering. This will be true of all series expansions in powers of ε. In (2), the displacements have been assumed to be arbitrarily large to within a rigid body translation. Rigid rotations have not been excluded by the assumed scaling (2). A prescribed displacement on ∂V_{0u} is assumed to be the same order as a displacement in the interior. This is a necessary requirement if the displacement is to be continuous throughout the closure of V_0.

The stresses and loads will be scaled by the load parameter τ and will be assumed to have the following asymptotic expansions:

$$\begin{aligned} S_{ij} &= \tau(S_{ij}^0+\varepsilon S_{ij}^1+\varepsilon^2 S_{ij}^2+\cdots) \\ \hat{t}_i &= \tau(\hat{t}_i^0+\varepsilon\hat{t}_i^1+\varepsilon^2\hat{t}_i^2+\cdots) \\ \hat{p}_i &= \tau(\hat{p}_i^0+\varepsilon\hat{p}_i^1+\varepsilon^2\hat{p}_i^2+\cdots) \\ f_i &= \frac{\tau}{L}(f_i^0+\varepsilon f_i^1+\varepsilon^2 f_i^2+\cdots) \\ S_{ij}^m &= O(1),\quad \hat{t}_i^m = O(1),\quad \hat{p}_i^m = O(1),\quad f_i^m = O(1),\quad m = 0,1,2,\ldots \end{aligned} \tag{3}$$

The asymptotic series expansion for the surface tractions must start at the order indicated so the stresses are continuous throughout the closure of V_0. Very large surface tractions cannot be reconciled with very small interior stresses, and still have the interior stresses appropriately continuous.

Finally, the key assumption is made: the strains are small,

$$\begin{aligned} E_{ij} &= \varepsilon E_{ij}^1+\varepsilon^2 E_{ij}^2+\cdots \\ E_{ij}^m &= O(1),\quad m = 1,2,3,\ldots \end{aligned} \tag{4}$$

Note that all strain components are scaled the same. In particular, the transverse shear strains are assumed initially to be of the same order as the in-plane strains and the transverse normal strain. This is a very important point: the displacements or strains are not required to satisfy any particular relationships, other than magnitude scalings.

The first variation of Π in (1) is required to vanish. Consider the first variation of the strain energy term

$$\delta\int_{V_0} W(E)\,\mathrm{d}V = \int_{V_0}\frac{\partial W}{\partial E_{ij}}(\varepsilon\delta E_{ij}^1+\varepsilon^2\delta E_{ij}^2+\cdots)\tau\varepsilon L^3\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3. \tag{5}$$

The partial derivatives in (5) will, in general, depend on E, and consequently on all of the strain functions defined in (4). Next, we must assume that these partial derivatives can be written

$$\frac{\partial W}{\partial E_{ij}} = \tau(\Omega_{ij}^0+\varepsilon\Omega_{ij}^1+\cdots),\quad \Omega_{ij}^m = O(1),\quad m = 0,1,2,\ldots \tag{6}$$

Clearly, this form can be used for linear elasticity if the elastic modulus is an order larger than the load scale. The nonlinear elastic consitutive relations in Johnson and Urbanik (1984) are also of this form.

The variation of Π can be written as an asymptotic series in ascending powers of ε using (2)–(6):

$$\begin{aligned} \delta\Pi &= \varepsilon^{-1}\delta\Pi^{-1}+\delta\Pi^0+\varepsilon^1\delta\Pi^1+\varepsilon^2\delta\Pi^2+\cdots = 0 \\ \delta\Pi^m &= O(\tau L^3),\quad m = -1,0,1,\ldots \end{aligned} \tag{7}$$

If (7) is satisfied approximately by taking only the first few terms in the sum, the result will be "near" the stationary point of Π. The deviation from the exact stationary point will be small, and will decrease (perhaps only up to a certain point) as more terms are included in the asymptotic series. Since (7) is a power series in ε, each individual term in the sum must vanish.

Consider the lowest order approximation to the stationary point

$$\delta\Pi^{-1} = \int_{V_0} \{\tfrac{1}{2}\delta S^0_{33} u^0_{i,3} u^0_{i,3} + S^0_{33} u^0_{i,3} \delta u^0_{i,3}\} \tau L^3 \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_3 = 0 \tag{8}$$

which in turn implies

$$\delta S^0_{33}\{u^0_{i,3} u^0_{i,3}\} = 0 \tag{9}$$

pointwise in V_0. Since the term in braces is a sum of three squares, each term in the sum must vanish independently

$$u^0_{i,3} = 0. \tag{10}$$

This can be integrated to give

$$u^0_i(\xi_j) = U^0_i(\xi_\alpha) \tag{11}$$

so the lowest order displacements are constant through the thickness.

The second term in (8) vanishes identically as a result of (10). It is interesting to note that $\delta\Pi^{-1}$ gives us no information about stresses in the interior. Further, there are no natural boundary conditions, equilibrium equations or constitutive equations. The Euler Lagrange equation (10) is in effect a strain-displacement relation.

Equations (10) or (11) are the simplest small strain plate theory. To obtain a higher order theory, consider the next term in (7):

$$\Pi^0 = \int_{V_0} \{\tfrac{1}{2} S^1_{33} u^0_{i,3} u^0_{i,3} + S^0_{33}(\delta_{i3} + u^1_{i,3}) u^0_{i,3} + S^0_{\alpha 3}(\delta_{i\alpha} + u^0_{i,\alpha}) u^0_{i,3}\} \tau L^3 \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_3 - \int_{\partial V_{0p}} u^0_i \hat{p}^0_i \tau L^3 \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2. \tag{12}$$

Taking variations, integrating by parts, and collecting terms results in

$$\delta S^1_{33}\{u^0_{i,3} u^0_{i,3}\} = 0 \quad \text{in } V_0 \tag{13}$$

$$\delta u^0_i\{(S^0_{\alpha 3}(\delta_{i\alpha} + u^0_{i,\alpha}) + S^0_{33}(\delta_{i3} + u^1_{i,3}))_{,3}\} = 0 \quad \text{in } V_0 \tag{14}$$

$$\delta u^0_i\{S^0_{\alpha 3}(\delta_{i\alpha} + u^0_{i,\alpha}) + S^0_{33}(\delta_{i3} + u^1_{i,3}) \mp \hat{p}^0_i\} = 0 \quad \text{on } \partial V_{0p\pm}. \tag{15}$$

Note the results of (13) are the same as those of (9). It is a characteristic of this method that all previous plate theories occur at any given order. The new results from $\delta\Pi^0$ are (14) and (15). The $\pm$ designation in (15) indicates a surface whose undeformed outward normal is in the positive or negative coordinate direction, respectively.

When (14) is integrated with boundary conditions (15), a condition on the pressures is obtained:

$$\hat{p}^0_i|_{\xi_3 = +(1/2)} = -\hat{p}^0_i|_{\xi_3 = -(1/2)}. \tag{16}$$

This implies that, in order to have a consistent theory, the loads must be "self-equilibrating". Because we are interested in more general loadings, in particular, nonself-equilibrating loads, we take

$$\hat{p}^0_i \equiv 0. \tag{17}$$

With this, a relation between transverse shear stresses and transverse normal stresses can be obtained in the form of an equilibrium equation from (14) and (15). Discussion of this relation will be postponed until it appears again in the next order theory.

LOWEST ORDER NONTRIVIAL PLATE THEORY

The next order theory will involve in-plane stresses, but will not have constitutive relations. It is, in essence, an inextensible membrane theory.

$$\begin{aligned}\Pi^1 = \int_{V_0} \{ & \tfrac{1}{2}S^2_{33}u^0_{i,3}u^0_{i,3} + S^1_{33}(\delta_{i3}+u^1_{i,3})u^0_{i,3} + \tfrac{1}{2}S^0_{33}((\delta_{i3}+u^1_{i,3})(\delta_{i3}+u^1_{i,3})-1) \\ & + S^1_{\alpha 3}(\delta_{i\alpha}+u^0_{i,\alpha})u^0_{i,3} + S^0_{\alpha 3}((\delta_{i\alpha}+u^0_{i\alpha})(\delta_{i3}+u^1_{i,3})+u^1_{i,\alpha}u^0_{i,3}) \\ & + \tfrac{1}{2}S^0_{\alpha\beta}((\delta_{i\alpha}+u^0_{i,\alpha})(\delta_{i\beta}+u^0_{i,\beta})-\delta_{\alpha\beta}) - f^0_i u^0_i\}\tau L^3\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \\ & - \int_{\partial V_{0u}} (u^0_i - \hat{u}^0_i)t^0_i\tau L^3\,\mathrm{d}\xi_{(t)}\,\mathrm{d}\xi_3 - \int_{\partial V_{0t}} u^0_i\hat{t}^0_i\tau L^3\,\mathrm{d}\xi_{(t)}\,\mathrm{d}\xi_3 \\ & - \int_{\partial V_{0p}} \{u^0_i\hat{p}^1_i + u^1_i\hat{p}^0_i\}\tau L^3\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \qquad (18)\end{aligned}$$

This form is more specific about the element of area for the prescribed traction and displacement integrals. The subscript t is to be treated as a constant, not a free index. It specifies the direction tangent to the coordinate line. Later, a subscript n will be used to indicate the direction normal to the area element. The subscripts t and n are the only exceptions to the index convention. They are never to be summed, and will always appear in parentheses.

Once again, we can take variations, integrate by parts and sort out the terms. The kinematic equations in V_0 at this order are (10) and

$$(\delta_{i3}+u^1_{i,3})(\delta_{i3}+u^1_{i,3})-1 = 0 \qquad (19)$$

$$(\delta_{i3}+u^1_{i,3})(\delta_{i\alpha}+u^0_{i,\alpha}) = 0 \qquad (20)$$

$$(\delta_{i\alpha}+u^0_{i,\alpha})(\delta_{i\beta}+u^0_{i,\beta})-\delta_{\alpha\beta} = 0. \qquad (21)$$

Equations (21) are the inextensibility conditions. Because the displacements have power series expansions in ε, there is also a power series expansion for the deformation gradient. Equations (19), (20) and (21) imply that the lowest order approximation to the deformation gradient is orthogonal. From (19) and (20) it can be shown (Berg and Johnson, 1989) that the variation of u^1_i through the thickness is linear in ξ_3,

$$u^1_i = U^1_i(\xi_\alpha)+\xi_3\bar{u}^1_i(\xi_\alpha) \qquad (22)$$

which is somewhat analogous to (11) in that it defines two new functions which depend only on in-plane coordinates.

The lowest order equilibrium equations at this order are the same as those from (14) and (15). Making use of (19) and (20) will simplify the relation between transverse shear and transverse normal stresses, in particular

$$S^0_{\alpha 3} = S^0_{33} = 0 \qquad (23)$$

where (17) has been used. The remaining equilibrium equations in the interior are

$$(S^0_{\alpha\beta}(\delta_{i\alpha}+U^0_{i,\alpha}))_{,\beta}+(S^1_{\alpha 3}(\delta_{i\alpha}+U^0_{i,\alpha})+S^1_{33}(\delta_{i3}+\bar{u}^1_i))_{,3}+f^0_i=0 \quad \text{in } V_0 \tag{24}$$

and the natural boundary conditions on the faces are

$$S^1_{\alpha 3}(\delta_{i\alpha}+U^0_{i,\alpha})+S^1_{33}(\delta_{i3}+\bar{u}^1_i)=\pm\hat{p}^1_i \quad \text{on } \xi_3=\pm\tfrac{1}{2} \tag{25}$$

where (11) and (22) have been used. At this point, it is necessary to integrate through the plate's thickness and introduce the stress resultants N, M and L:

$$N^m_{ij}=\int_{-1/2}^{1/2}S^m_{ij}\,\mathrm{d}\xi_3,\quad M^m_{ij}=\int_{-1/2}^{1/2}S^m_{ij}\xi_3\,\mathrm{d}\xi_3,\quad L^m_{ij}=\int_{-1/2}^{1/2}S^m_{ij}\frac{(\xi_3)^2}{2}\,\mathrm{d}\xi_3,\quad m=0,1,2,\ldots \tag{26}$$

The moment resultants M and L will be used later. The equilibrium equations (24) take the form

$$(N^0_{\alpha\beta}(\delta_{i\alpha}+U^0_{i,\alpha}))_{,\beta}+\gamma^0_i=0. \tag{27}$$

The body force and surface tractions have been combined into a single load variable

$$\gamma^m_i=\int_{-1/2}^{1/2}f^m_i\,\mathrm{d}\xi_3+\hat{p}^{m+1}_i|_{\xi_3=1/2}+\hat{p}^{m+1}_i|_{\xi_3=-1/2},\quad m=0,1,2,\ldots \tag{28}$$

It is interesting to note that surface tractions on the faces of the plate are indistinguishable from averaged body forces in this theory. This is usually assumed to be true in membrane theories, but is never rigorously justified. The natural boundary conditions are

$$N^0_{(n)\beta}(\delta_{i\beta}+U^0_{i,\beta})=\pm\int_{-1/2}^{1/2}\hat{t}^0_i\,\mathrm{d}\xi_3=n^0_i \quad \text{on } \partial V_{0t\pm}. \tag{29}$$

With (20) and (21), (29) can be rewritten

$$\begin{aligned} N^0_{(n)\beta}&=(\delta_{i\beta}+U^0_{i,\beta})n^0_i=\text{in-plane component of } n^0_i\\ 0&=(\delta_{i3}+\bar{u}^1_i)n^0_i=\text{transverse component of } n^0_i \end{aligned} \tag{30}$$

where in-plane and transverse mean tangent to the deformed plate and normal to the deformed plate, respectively. The fact that the transverse component of the applied traction in the deformed configuration is zero is not a restriction on the way the loads may be applied. Rather, it is a statement that the plate deforms in such a way that at the edges, the midsurface has rotated so that it is in the plane of the applied loads. Keep in mind that the length scale assumption prohibits loads which change significantly over distances shorter than $O(L)$. Lastly, the forced boundary conditions on edges where displacements are prescribed are

$$U^0_i=\hat{u}^0_i \quad \text{on } \partial V_{0u}. \tag{31}$$

This means that, in order to have a consistent theory, only a membrane-type displacement condition may be prescribed. Only the displacement on the midsurface may be specified.

Because the equilibrium equation only involves in-plane components, and the boundary conditions only allow in-plane components, this theory is truly a membrane theory. This theory is statically determinate. It is a special case of membrane shell theory for developable shells.

While the preceding plate theories are of some interest in their own right, they do not exhibit all the qualities normally associated with a plate theory. The next theory to be presented includes constitutive equations, strain–displacement relations, and both in-plane and moment stress resultants. Further, nonzero transverse shear stresses will play a role. The variational principle for this theory is

$$\begin{aligned}\Pi^2 = \int_{V_0} \{ & \text{strain energy term} - S^0_{ij}E^0_{ij} + \tfrac{1}{2}S^3_{33}u^0_{i,3}u^0_{i,3} + S^2_{33}(\delta_{i3}+u^1_{i,3})u^0_{i,3} \\
& + S^1_{33}(\tfrac{1}{2}((\delta_{i3}+u^1_{i3})(\delta_{i3}+u^1_{i,3})-1)+u^2_{i,3}u^0_{i,3}) + S^0_{33}((\delta_{i3}+u^1_{i,3})u^2_{i,3}+u^3_{i,3}u^0_{i,3}) \\
& + S^2_{\alpha 3}(\delta_{i\alpha}+u^0_{i,\alpha})u^0_{i,3} + S^1_{\alpha 3}((\delta_{i\alpha}+u^0_{i,\alpha})(\delta_{i3}+u^1_{i,3})+u^1_{i,\alpha}u^0_{i,3}) \\
& + S^0_{\alpha 3}((\delta_{i\alpha}+u^0_{i,\alpha})u^2_{i,3}+(\delta_{i3}+u^1_{i,3})u^1_{i,\alpha}+u^2_{i,\alpha}u^0_{i,3}) \\
& + \tfrac{1}{2}S^1_{\alpha\beta}((\delta_{i\alpha}+u^0_{i,\alpha})(\delta_{i\beta}+u^0_{i,\beta})-\delta_{\alpha\beta}) + S^0_{\alpha\beta}(\delta_{i\alpha}+u^0_{i,\alpha})u^1_{i,\beta} \\
& - f^0_i u^1_i - f^1_i u^0_i\}\tau L^3\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 - \int_{\partial V_{0u}} \{(u^0_i-\hat{u}^0_i)t^1_i + (u^1_i-\hat{u}^1_i)t^0_i\}\tau L^3\,\mathrm{d}\xi_{(t)}\,\mathrm{d}\xi_3 \\
& - \int_{\partial V_{0t}} \{u^0_i\hat{t}^1_i + u^1_i\hat{t}^0_i\}\tau L^3\,\mathrm{d}\xi_{(t)}\,\mathrm{d}\xi_3 - \int_{\partial V_{0p}} \{u^0_i\hat{p}^2_i + u^1_i\hat{p}^1_i + u^2_i\hat{p}^0_i\}\tau L^3\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \end{aligned} \tag{32}$$

We have not been too specific about the "strain energy term" because we do not know how to write it down directly. We do, however, know its variation from (5) and (6). If the variation of (32) is set equal to zero, and integrated by parts, the governing equations and boundary conditions of this theory are obtained.

The kinematic equations of this theory are (10), (19), (20) and (21) and the strain–displacement equations

$$E^1_{33} = (\delta_{i3}+\bar{u}^1_i)u^2_{i,3} \tag{33}$$

$$2E^1_{\alpha 3} = (\delta_{i\alpha}+U^0_{i,\alpha})u^2_{i,3} + (\delta_{i3}+\bar{u}^1_i)U^1_{i,\alpha} + \xi_3(\delta_{i3}+\bar{u}^1_i)\bar{u}^1_{i,\alpha} \tag{34}$$

$$E^1_{\alpha\beta} = (\delta_{i\alpha}+U^0_{i,\alpha})U^1_{i,\beta} + \xi_3(\delta_{i\alpha}+U^0_{i,\alpha})\bar{u}^1_{i,\beta} = e^1_{\alpha\beta} + \xi_3\kappa^1_{\alpha\beta} \tag{35}$$

where the notation has been simplified by making an implied symmetry assumption on the strain tensor as well as using (11) and (22). In (35), in-plane membrane strains and bending strains have also been identified.

The constitutive equations from the variation of (32) put limits on the form the strain energy function can assume. The transverse normal and transverse shear terms are

$$\Omega^0_{33} = S^0_{33} = 0 \tag{36}$$

$$\Omega^0_{\alpha 3} = S^0_{\alpha 3} = 0 \tag{37}$$

where (17) and (23) have been used. The in-plane terms are

$$\Omega^0_{\alpha\beta} = S^0_{\alpha\beta} \tag{38a}$$

$$N^0_{\alpha\beta} = \int_{-1/2}^{1/2} \Omega^0_{\alpha\beta}\,\mathrm{d}\xi_3, \quad M^0_{\alpha\beta} = \int_{-1/2}^{1/2} \Omega^0_{\alpha\beta}\xi_3\,\mathrm{d}\xi_3, \quad L^0_{\alpha\beta} = \int_{-1/2}^{1/2} \Omega^0_{\alpha\beta}\frac{\xi_3^2}{2}\,\mathrm{d}\xi_3. \tag{38b}$$

Additional higher order terms could be defined, if required. Thus the loading assumption and strain scaling affect the form the constitutive equations may take. The equilibrium equations and natural boundary conditions of this theory are

$$(S^0_{\alpha\beta}(\delta_{i\beta}+U^0_{i,\beta}))_{,\alpha} + (S^1_{\alpha 3}(\delta_{i\alpha}+U^0_{i,\alpha}) + S^1_{33}(\delta_{i3}+\bar{u}^1_i))_{,3} + f^0_i = 0 \quad \text{in } V_0 \tag{39}$$

$$(S^1_{\alpha\beta}(\delta_{i\beta}+U^0_{i,\beta})+S^1_{\alpha3}(\delta_{i3}+\bar{u}^1_i)+S^0_{\alpha\beta}u^1_{i,\beta})_{,\alpha}+(S^2_{33}(\delta_{i3}+\bar{u}^1_i)+S^2_{\alpha3}(\delta_{i\alpha}+U^0_{i,\alpha})$$
$$+S^1_{33}u^2_{i,3}+S^1_{\alpha3}u^1_{i,\alpha})_{,3}+f^1_i=0 \quad \text{in } V_0 \quad (40)$$

$$S^1_{\alpha3}(\delta_{i\alpha}+U^0_{i,\alpha})+S^1_{33}(\delta_{i3}+\bar{u}^1_i)=\pm\hat{p}^1_i \quad \text{on } \xi_3=\pm\tfrac{1}{2} \quad (41)$$

$$S^2_{33}(\delta_{i3}+\bar{u}^1_i)+S^2_{\alpha3}(\delta_{i\alpha}+U^0_{i,\alpha})+S^1_{33}u^2_{i,3}+S^1_{\alpha3}u^1_{i,\alpha}=\pm\hat{p}^2_i \quad \text{on } \xi_3=\pm\tfrac{1}{2}. \quad (42)$$

With (41), (39) can be integrated through the thickness to get (27) as the lowest order membrane equilibrium equation at this order. With (42), (40) can be integrated to give

$$(N^1_{\alpha\beta}(\delta_{i\beta}+U^0_{i,\beta})+N^1_{\alpha3}(\delta_{i3}+\bar{u}^1_i)+N^0_{\alpha\beta}U^1_{i,\beta}+M^0_{\alpha\beta}\bar{u}^1_{i,\beta})_{,\alpha}+\gamma^1_i=0. \quad (43)$$

A moment equilibrium equation can be obtained by multiplying (39) by ξ_3 and integrating through the thickness:

$$(M^0_{\alpha\beta}(\delta_{i\beta}+U^0_{i,\beta}))_{,\alpha}-N^1_{33}(\delta_{i3}+\bar{u}^1_i)-N^1_{\beta3}(\delta_{i\beta}+U^0_{i,\beta})+\bar{\gamma}^0_i=0 \quad (44)$$

where, once again, the body force and surface traction have been combined,

$$\bar{\gamma}^m_i=\int_{-1/2}^{1/2}\xi_3 f^m_i\,\mathrm{d}\xi_3+\tfrac{1}{2}\hat{p}^{m+1}_i|_{\xi_3=1/2}-\tfrac{1}{2}\hat{p}^{m+1}_i|_{\xi_3=-1/2} \quad m=0,1,2,\ldots \quad (45)$$

Equations (27), (43) and (44) are written in terms of components in the reference (undeformed) configuration. Since the lowest order deformation gradient is orthogonal, the equilibrium equations can easily be written in terms of components referred to the (lowest order approximation to the) deformed basis. The three equations in (27) become

$$N^0_{\alpha\beta,\alpha}+\gamma^0_i(\delta_{i\beta}+U^0_{i,\beta})=0$$
$$N^0_{\alpha\beta,\alpha}\kappa^1_{\alpha\beta}=\gamma^0_i(\delta_{i3}+\bar{u}^1_i). \quad (46)$$

The three equations in (44) become

$$M^0_{\alpha\beta,\beta}+\bar{\gamma}^0_i(\delta_{i\alpha}+U^0_{i,\alpha})=N^1_{\alpha3}$$
$$M^0_{\alpha\beta}\kappa^1_{\alpha\beta}+N^1_{33}=\bar{\gamma}^0_i(\delta_{i3}+\bar{u}^1_i) \quad (47)$$

and the three equations in (43) become

$$N^1_{\alpha\beta,\beta}+N^1_{\beta3}\kappa^1_{\alpha\beta}+M^0_{\beta\gamma,\gamma}\kappa^1_{\alpha\beta}+M^0_{\alpha\gamma}\kappa^1_{\beta\gamma,\beta}+((N^0_{\beta\gamma}U^1_{i,\beta})_{,\gamma}+\gamma^0_i)(\delta_{i\alpha}+U^0_{i,\alpha})=0$$
$$N^1_{\alpha3,\alpha}-N^1_{\alpha\beta}\kappa^1_{\alpha\beta}-M^0_{\alpha\beta}\kappa^1_{\alpha\gamma}\kappa^1_{\beta\gamma}+((N^0_{\beta\gamma}U^1_{i,\beta})_{,\gamma}+\gamma^0_i)(\delta_{i3}+\bar{u}^0_i)=0. \quad (48)$$

In applications, these convected forms of the equilibrium equations are sometimes easier to use than the forms given earlier. Equations (30) were also written in terms of the convected basis.

The natural boundary conditions associated with this theory are (30) and

$$N^1_{(n)\alpha}+M^0_{(n)\beta}\kappa^1_{\alpha\beta}=n^1_i(\delta_{i\alpha}+U^0_{i,\alpha})-n^0_i e^1_{\alpha\beta}(\delta_{i\beta}+U^0_{i,\beta}) \quad \text{on } \partial V_{0t} \quad (49)$$

where (30) has been used.† This type of boundary condition is obtained in ordinary shell theory where the curvatures are specified and the reference configuration is not necessarily flat (Niordson, 1985).

† There is an error in equation (8.2) in Berg and Johnson (1989) where the factor of 2 should not be present, as in (49).

In (49), only the in-plane force boundary condition is given. There are also two moment boundary conditions which could be obtained directly from the variation of (32), but this will not result in Kirchhoff-type conditions. In finite deformation plate theory (von Karman theory), the transverse shear condition is derived from an energy (minimum) principle, rather than a stationary principle. To obtain Kirchhoff-type transverse shear conditions in the large deformation nonlinear theory, it is necessary to substitute the known kinematics into the Hu–Washizu variational principle. In effect, this converts the Hu–Washizu principle to a principle with constaints (strain–displacement relations) imposed on the fields. The resulting principle does not apparently have a name associated with it, but is listed as Type VI by Oden and Reddy (1976):

$$\Pi^2_{\mathrm{VI}} = \int_{V_0} \{S^0_{\alpha\beta}(\delta_{i\alpha}+U^0_{i,\alpha})U^1_{i,\beta}+\xi_3 S^0_{\alpha\beta}(\delta_{i\alpha}+U^0_{i,\alpha})\bar{u}^1_{i,\beta}-f^0_i u^1_i-f^1_i u^0_i\}\tau L^3\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3$$
$$-\int_{\partial V_{0t}} \{u^0_i \bar{t}^1_i+u^1_i \bar{t}^0_i\}\tau L^3\,\mathrm{d}\xi_{(t)}\,\mathrm{d}\xi_3. \quad (50)$$

When variations of (50) are taken, some care must be used in order that only independent variations of independent functions are taken. From (19), (20) and (21), it can be shown by direct substitution that there are relations between U^0_i and $\bar{u}^1_i$:

$$(\delta_{i3}+\bar{u}^1_i) = \varepsilon_{ijk}(\delta_{j1}+U^0_{j,1})(\delta_{k2}+U^0_{k,2})$$
$$(\delta_{i1}+U^0_{i1}) = \varepsilon_{ijk}(\delta_{j2}+U^0_{j,2})(\delta_{k3}+\bar{u}^1_k)$$
$$(\delta_{i2}+U^0_{i,2}) = \varepsilon_{ijk}(\delta_{j3}+\bar{u}^1_j)(\delta_{k1}+U^0_{k,1}). \quad (51)$$

These follow from the definition of the convected base vectors. Assume $\mathbf{e}_i$ are a set of orthonormal base vectors aligned with the coordinate axes in the reference configuration. These base vectors will be mapped into a set of convected base vectors $\mathbf{h}_i$. Since the lowest order deformation gradient is orthogonal, the lowest order approximation to the convected base vectors is the orthonormal set $\mathbf{h}^0_i$. The two sets of orthonormal base vectors are related as follows:

$$\mathbf{e}_i = (\delta_{i\alpha}+U^0_{i,\alpha})\mathbf{h}^0_\alpha+(\delta_{i3}+\bar{u}^1_i)\mathbf{h}^0_3$$
$$\mathbf{h}^0_\alpha = (\delta_{i\alpha}+U^0_{i,\alpha})\mathbf{e}_i$$
$$\mathbf{h}^0_3 = (\delta_{i3}+\bar{u}^1_i)\mathbf{e}_i. \quad (52)$$

Because the convected base vectors (at the lowest approximation) are orthonormal, any one can be written in terms of the other two via the cross product. It is these cross product relations that are shown in (51). Π^2_{VI} can be written in terms of only two independent displacement functions using (51):

$$\Pi^2_{\mathrm{VI}} = \int_{V_0} \{S^0_{\alpha\beta}(\delta_{i\alpha}+U^0_{i,\alpha})U^1_{i,\beta}+\xi_3 S^0_{\alpha\beta}(\delta_{i\alpha}+U^0_{i,\alpha})(\varepsilon_{ijk}(\delta_{j1}+U^0_{j,1})(\delta_{k2}+U^0_{k,2}))_{,\beta}$$
$$-(U^1_i+\xi_3\varepsilon_{ijk}(\delta_{j1}+U^0_{j,1})(\delta_{k2}+U^0_{k,2}))f^0_i-U^0_i f^1_i\}\tau L^3\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3$$
$$-\int_{\partial V_{0t}} \{(U^1_i+\xi_3\varepsilon_{ijk}(\delta_{j1}+U^0_{j,1})(\delta_{k2}+U^0_{k,2}))\bar{t}^0_i+U^0_i\bar{t}^1_i\}\tau L^3\,\mathrm{d}\xi_{(t)}\,\mathrm{d}\xi_3. \quad (53)$$

Note that, unlike Π^2, the new principle Π^2_{VI} involves two derivatives of the displacements. When the variation of this principle is integrated by parts, there will be derivatives of both displacement variations and stresses in the boundary integral terms. Additionally, the higher order derivatives in the principle lead to conditions on the twist moment at the corner.

When the first variation of (53) is integrated by parts, and terms are collected, the following natural boundary conditions are obtained:

$$M^0_{(nn)} = \int_{-1/2}^{1/2} \xi_3(\hat{\mathbf{t}}^1 \cdot \mathbf{h}^0_{(n)})\, \mathrm{d}\xi_3 \tag{54}$$

$$M^0_{(nt,t)} + N^1_{(n)3} = \int_{-1/2}^{1/2} (\hat{\mathbf{t}}^1 \cdot \mathbf{h}^0_3) + \xi_3(\hat{\mathbf{t}}^1_{,(t)} \cdot \mathbf{h}^0_{(t)})\, \mathrm{d}\xi_3 - N^0_{\alpha(n)} U^1_{i,\alpha}(\delta_{i3} + \bar{u}^1_i). \tag{55}$$

These are moment boundary conditions analogous to the Kirchhoff natural boundary conditions of ordinary linear plate theory. Finally, there is a corner condition:

$$2M^0_{12} + \int_{-1/2}^{1/2} \xi_3(\hat{\mathbf{t}}^1 \cdot \mathbf{h}^0_1 + \hat{\mathbf{t}}^1 \cdot \mathbf{h}^0_2)\, \mathrm{d}\xi_3 + \hat{R}^0_i(\delta_{i3} + \bar{u}^1_i) = 0. \tag{56}$$

Clearly, we should have anticipated that there could have been a concentrated corner force, and included it in the potential energy. The variational principle can be modified to include this term:

$$\Pi^2 \to \Pi^2 - \hat{R}^0_i u^0_i. \tag{57}$$

A VON KARMAN-TYPE THEORY

The next term in the variational principle sum can be written in a straightforward way. The resulting strain-displacement equations are (10), (19)-(22), (33)-(35) and

$$E^2_{33} = (\delta_{i3} + \bar{u}^1_i)u^3_{i,3} + \tfrac{1}{2}u^2_{i,3}u^2_{i,3} \tag{58}$$

$$2E^2_{\alpha 3} = (\delta_{i3} + \bar{u}^1_i)u^2_{i,\alpha} + (\delta_{i\alpha} + U^0_{i,\alpha})u^3_{i,3} + u^1_{i,\alpha}u^2_{i,3} \tag{59}$$

$$E^2_{\alpha\beta} = (\delta_{i\alpha} + U^0_{i,\alpha})u^2_{i,\beta} + \tfrac{1}{2}u^1_{i,\alpha}u^1_{i,\beta}. \tag{60}$$

The constitutive equations are (36)–(38) and

$$\Omega^1_{ij} = S^1_{ij} \tag{61}$$

where, once again, (17) and (23) are used. For simplicity, the material is assumed to be linear elastic from this point onward. The linear elastic material assumption, along with (23), gives

$$E^1_{\alpha 3} = 0 \tag{62}$$

and further implies that E^1_{33} is a linear combination of E^1_{11} and E^1_{22}. With this, analogous to (11) and (22), the ξ_3 variation of u^2_i is known:

$$u^2_i = U^2_i + \xi_3 \bar{u}^2_i + \frac{(\xi_3)^2}{2}\bar{\bar{u}}^2_i. \tag{63}$$

Using (63), the ξ_3 dependence in $E^2_{\alpha\beta}$ can be stated explicitly:

$$E^2_{\alpha\beta} = \{(\delta_{i\alpha}+U^0_{i,\alpha})U^2_{i,\beta}+\tfrac{1}{2}U^1_{i,\alpha}U^1_{i,\beta}\}+\xi_3\{(\delta_{i\alpha}+U^0_{i,\alpha})\bar{u}^2_{i,\beta}+U^1_{i,\alpha}\bar{u}^1_{i,\beta}\}$$
$$+\frac{(\xi_3)^2}{2}\{(\delta_{i\alpha}+U^0_{i,\alpha})\bar{\bar{u}}^2_{i,\beta}+\tfrac{1}{2}\bar{u}^1_{i,\alpha}\bar{u}^1_{i,\beta}\} = e^2_{\alpha\beta}+\xi_3\kappa^2_{\alpha\beta}+\frac{(\xi_3)^2}{2}\lambda^2_{\alpha\beta}. \quad (64)$$

The equilibrium equations associated with theory are (27), (43), (44) and

$$(N^2_{\alpha 3}(\delta_{i3}+\bar{u}^1_i)+N^1_{\alpha 3}\bar{u}^2_i+M^1_{\alpha 3}\bar{\bar{u}}^2_i+N^2_{\alpha\beta}(\delta_{i\beta}+U^0_{i,\beta})+N^1_{\alpha\beta}U^1_{i,\beta}+M^1_{\alpha\beta}\bar{u}^1_{i,\beta}+N^0_{\alpha\beta}U^2_{i,\beta}$$
$$+M^0_{\alpha\beta}\bar{u}^2_{i,\beta}+L^0_{\alpha\beta}\bar{\bar{u}}^2_{i,\beta})_{,\alpha}+\gamma^2_i = 0 \quad (65)$$

which is an in-plane equilibrium equation,

$$(M^1_{\alpha 3}(\delta_{i3}+\bar{u}^1_i)+M^1_{\alpha\beta}(\delta_{i\beta}+U^0_{i,\beta})+M^0_{\alpha\beta}U^1_{i,\beta}+L^0_{\alpha\beta}\bar{u}^1_{i,\beta})_{,\alpha}-N^2_{33}(\delta_{i3}+\bar{u}^1_i)-N^2_{\alpha 3}(\delta_{i\alpha}+U^0_{i,\alpha})$$
$$-N^1_{33}\bar{u}^2_i-M^1_{33}\bar{\bar{u}}^2_i-N^1_{\alpha 3}U^1_{i,\alpha}-M^1_{\alpha 3}\bar{u}^1_{i,\alpha}+\bar{\gamma}^1_i = 0 \quad (66)$$

which is a first moment equation obtained from (40), and

$$(L^0_{\alpha\beta}(\delta_{i\beta}+U^0_{i,\beta}))_{,\alpha}-M^1_{33}(\delta_{i3}+\bar{u}^1_i)-M^1_{\alpha 3}(\delta_{i\alpha}+U^0_{i,\alpha})+\bar{\bar{\gamma}}^0_i = 0, \quad (67)$$

which is a second moment equation obtained from (25), where

$$\bar{\bar{\gamma}}^0_i = \int_{-1/2}^{1/2}\frac{(\xi_3)^2}{2}f^0_i\,\mathrm{d}\xi_3+\tfrac{1}{8}\hat{p}^1_i|_{\xi_3=1/2}+\tfrac{1}{8}\hat{p}^1_i|_{\xi_3=-1/2}. \quad (68)$$

Natural boundary conditions also follow directly from the variational principle. Each successive theory will involve higher order moments of the stresses, and associated higher order kinematics. Presumably, additional higher order theories could be derived.

It is straightforward to show that this theory is a generalization of the von Karman theory by deriving it as a special case. To obtain the von Karman equations, a new strain scaling is needed to replace (4). The appropriate strain scaling is

$$E_{ij} = O(\varepsilon^2), \quad (69)$$

which implies

$$e^1_{ij} = \kappa^1_{ij} = 0. \quad (70)$$

If the curvatures vanish to lowest order, the plate must remain nearly flat and can only undergo, to lowest order, rigid body rotation and translation. Assume the boundary conditions prohibit rigid rotation and translation, and consequently take

$$U^0_i = 0 \quad (71)$$

which, using (51), implies

$$\bar{u}^1_i = 0. \quad (72)$$

These, along with $e^1_{ij} = 0$ give

$$U^1_{\alpha,\beta}+U^1_{\beta,\alpha} = 0 \quad (73)$$

where the symmetry implied in (35) has been written explicitly. Equation (73) can be integrated to give

$$U_\alpha^1 = 0 \tag{74}$$

where, once again, the boundary conditions are assumed to prohibit any rigid body motion.

Equations (33) and (34), along with (20), give

$$u_{3,3}^2 = 0, \quad u_\alpha^2 = U_\alpha^2 - \xi_3 U_{3,\alpha}^1 \tag{75}$$

with which (64) can be written

$$2e_{\alpha\beta}^2 = U_{\alpha,\beta}^2 + U_{\beta,\alpha}^2 + U_{3,\alpha}^1 U_{3,\beta}^1 \tag{76}$$

$$\kappa_{\alpha\beta}^2 = -U_{3,\alpha\beta}^1 \tag{77}$$

$$\lambda_{\alpha\beta}^2 = 0. \tag{78}$$

Novozhilov (1953) derived a theory similar to this, but was forced to make the *ad hoc* assumption that (78) was true. It is now clear that (78) proceeds naturally from the strain scalings. Equations (76) and (77) are the kinematics assumed in the von Karman theory. It is now possible to see precisely under what conditions those assumptions are valid.

The constitutive equations (38) will imply that

$$N_{\alpha\beta}^0 = 0 \tag{79}$$

which in turn, using the equilibrium equation (27), gives

$$\gamma_i^0 = 0. \tag{80}$$

This places an additional restriction on the loads that can be applied. The remaining equilibrium equations become

$$N_{i\alpha,\alpha}^1 + \gamma_i^1 = 0 \tag{81}$$

and

$$M_{\alpha\beta,\beta}^1 + \bar{\gamma}_\alpha^1 = N_{\alpha 3}^1, \quad \gamma_3^1 = N_{33}^1. \tag{82}$$

These equilibrium equations, along with (76) and (77), form the basic equations of the von Karman theory. Equations (65) and (66) for higher order stress resultants are not required in order to have a consistent theory. A final point worth noting is that it is indeed consistent to keep $(82)_1$ when calculating, for example, the maximum principle stress, as in a simple failure theory.

EXAMPLES

To further demonstrate the nature of these theories, several examples will be presented. The first is an example of the lowest order nontrivial theory whose governing equations are (21) and (27). Let the plate occupy the following domain

$$-\pi R \leqslant X_1 \leqslant \pi R = \pi L, \quad -\infty \leqslant X_2 \leqslant \infty, \quad -\frac{h}{2} \leqslant X_3 \leqslant \frac{h}{2}$$

and take the loads to be

$$\gamma_1^0 = \sin\xi_1, \quad \gamma_2^0 = 0, \quad \gamma_3^0 = -\cos\xi_1 \tag{83}$$

with the natural boundary conditions

$$n_1^0 = \cos\xi_1, \quad n_2^0 = 0, \quad n_3^0 = \sin\xi_1 \quad \text{on } X_1 = \pm\pi R. \tag{84}$$

By substituting into (21) and (27), it can be shown that

$$\begin{aligned} &N_{11}^0 = 1, \quad N_{12}^0 = N_{22}^0 = 0 \\ &U_1^0 = -\xi_1 + \sin\xi_1, \quad U_2^0 = 0, \quad U_3^0 = 1-\cos\xi_1 \end{aligned} \tag{85}$$

is a solution to (21) and (27) satisfying the boundary condition (30). This is a case of a very long flat plate rolled into a cylinder with radius R. The internal pressure is

$$p = \sqrt{\hat{p}_i\hat{p}_i} = (1+O(\varepsilon))\tau\varepsilon\sqrt{\gamma_i^0\gamma_i^0} = (1+O(\varepsilon))\tau\varepsilon. \tag{86}$$

Note that the internal stresses are $O(1)$ while the applied pressures are $O(\varepsilon)$. If

$$\sigma h = \int_{-h/2}^{h/2} S_{11}\,\mathrm{d}X_3 \tag{87}$$

defines the mean stress σ, then (85) and (86) give

$$\sigma = (1+O(\varepsilon))p\frac{R}{h} \tag{88}$$

which is exactly the elementary strength of materials result, with an estimate of the error.

As a second example using the same theory consider lifting a very large thin rectangular plate by one corner using a concentrated force at the corner, with gravity acting normal to the undeformed plate. Since the plate is very large, and for suitable loads, only the region near the corner where the loads are applied will be bent. The remainder will remain flat, presumably lying on a flat surface. If the X_1- and X_2-axis are along two free edges of the large flat plate occupying the first quadrant in the X_1–X_2 plane and the concentrated load applied at the origin is

$$P\mathbf{e}_3 - \frac{Q}{\sqrt{2}}(\mathbf{e}_1+\mathbf{e}_2) \tag{89}$$

the deformation will be symmetric in X_1 and X_2. It is not difficult to show that the displacement gradients must be

$$\begin{aligned} &U_{1,1}^0 = U_{1,2}^0 = U_{2,1}^0 = U_{2,2}^0 = \tfrac{1}{2}\bar{u}_3^1 = -\tfrac{1}{2}(1-\cos\theta) \\ &U_{3,1}^0 = U_{3,2}^{(0)} = -\bar{u}_1^1 = -\bar{u}_2^1 = -\frac{1}{\sqrt{2}}\sin\theta \end{aligned} \tag{90}$$

where $\theta(X_1, X_2)$ is the rotation a given generator has undergone. These are easy to derive using the finite rotation vector representation of the deformation gradient as in Berg (1988). The weight per unit area of the plate will be γ and will act in the minus X_3-direction. The curvature components can be shown to be

$$\kappa^1_{11} = \kappa^1_{21} = \frac{1}{\sqrt{2}}\theta_{,1}, \quad \kappa^1_{12} = \kappa^1_{22} = \frac{1}{\sqrt{2}}\theta_{,2} \tag{91}$$

and since the curvature tensor must be symmetric,

$$\theta = \theta(s) = \theta\left(\frac{\xi_1+\xi_2}{\sqrt{2}}\right), \quad \theta' = \frac{\partial\theta}{\partial s} \tag{92}$$

so

$$\kappa^1_{\alpha\beta} = \tfrac{1}{2}\theta'. \tag{93}$$

If a t-axis is defined in the direction perpendicular to the s-axis, the new curvature components are

$$\kappa^1_{tt} = \kappa^1_{st} = \kappa^1_{ts} = 0, \quad \kappa_{ss} = \theta'. \tag{94}$$

The equilibrium equations can be easily integrated subject to the condition that the plate is stress-free on $X_1 = 0$ and $X_2 = 0$. This results in an equation for $\theta(s)$

$$s\theta'' \cos\theta - \theta' \cos\theta + 2s(\theta')^2 \sin\theta = 0 \tag{95}$$

which is singular at the corner of the plate, $s = 0$. It is perhaps possible to solve (95) directly, but it is more straightforward to consider the net total in-plane (horizontal) force in the deformed plate

$$H(s) = \cos\theta \int_{-s}^{s} N^0_{ss}(s,t)\,\mathrm{d}t = -2s\gamma \frac{\cos^2\theta(s)}{\theta'(s)}. \tag{96}$$

It is easy to show that (95) implies that the horizontal force is independent of s, subject to the conditions that the curvature and N^0_{ss} do not vanish. The horizontal force is known from (89)

$$H(s) = Q \tag{97}$$

so (96) can be integrated to give

$$\theta(s) = \tan^{-1}\left\{\frac{P-\gamma s^2}{Q}\right\}. \tag{98}$$

With this, in practice, all the stresses and deflections can be found. Note that when $s = \sqrt{P/\gamma}$, $\theta(s)$ must vanish. This is the point where the vertical concentrated force balances the weight of the bent corner of the plate. At the point where $\theta(s)$ vanishes, the horizontal load does not vanish, and is presumed to be balanced by friction forces between the very large unbent part of the plate and its supporting surface. This is a two-dimensional analog of the problem of lifting a string off a flat surface by a concentrated force at its end.

As a final example, consider the special case of the theory derived from (32) when $u_2 = 0$. This problem was solved by Johnson (1985) where he shows that under this plate-strain condition, (19)–(21) are satisfied by

$$\begin{aligned} 1+U^0_{1,1} &= \cos\theta(\xi_1), \qquad & U^0_{3,1} &= \sin\theta(\xi_1) \\ \bar{u}^1_1 &= -\sin\theta(\xi_1), & 1+\bar{u}^1_3 &= \cos\theta(\xi_1) \end{aligned} \tag{99}$$

where the variable θ is not the same as was used in the previous examples. We will also assume that all deformation and stress variables are independent of ξ_2. From (99) the curvature can be calculated, then substituted into (46)

$$N^0_{11}\theta_{,1} = 0 \tag{100}$$

which says either the lowest order normal stress vanishes, or the plate remains flat. Assume the former is true. By combining the in-plane equations from (43), and integrating, the transverse shear is

$$N^1_{13} = a_1 \sin\theta + a_2 \cos\theta \tag{101}$$

where a_1 and a_2 are constants. Compressive axial dead loads are to be applied to the plate's edges, so

$$\hat{t}_2 = \hat{t}_3 = 0, \quad \int_{-1/2}^{1/2} \hat{t}_1 \, \mathrm{d}\xi_3 = -P \tag{102}$$

where P is the dimensionless axial load per unit length. Equation (55) then gives

$$N^1_{13}|_{\text{end}} = P\sin\theta. \tag{103}$$

With this, the moment equilibrium equation (47), along with (101), gives

$$M^0_{11,1} = P\sin\theta. \tag{104}$$

Finally, for a linear elastic material

$$M^0_{11} = -D\theta_{,1} \tag{105}$$

and (104) becomes

$$\theta_{,11} + \frac{P}{D}\sin\theta = 0 \tag{106}$$

which is the equation of the elastica.

It is interesting to note that the elastica is a special case of a theory with the strain scaling (4). To arrive at the von Karman equations, the more restrictive strain scaling (69) is required. Simmonds (1979) comments that "it is a well-known deficiency of the von Karman equations that they do not contain as a special case the equations of the elastica". It is not so much a defect, but rather inappropriate to expect the von Karman equations to govern the elastica. To arrive at the von Karman equations, the assumption which neglects motions akin to those of the elastica is made. This assumption is subtly included in the von Karman equations' assumed strain–displacement relation. This demonstrates the real power of the asymptotic integration method. The assumptions serve to enlighten rather than to obscure.

DISCUSSION

It is possible to derive the governing equations of the various theories presented by simply studying the equations of three-dimensional nonlinear elasticity. The advantages of using the variational formulation are that the natural boundary conditions are also derived, and that there is a natural ordering to the importance of the theories. When starting with the governing differential equations, it is not always clear precisely which equations are required in any given theory. When working with the variational principle, on the other hand, the grouping of equations into theories occurs quite naturally. For example, it is not initially obvious that it is completely consistent to have a theory with stresses but not have any constitutive equations or strains. On the other hand, membrane shell theories are derived by making just these assumptions. This highlights the importance of this method.

One is able to derive the equations governing the theory based on a reasonable geometric scaling assumption and an assumption on the material behavior. The limitation to small strains is not too restrictive since most engineering materials except elastomers cannot undergo elastic strains of any appreciable magnitude.

Aside from the two assumptions discussed above, there is an additional implicit assumption on the material behavior. By not introducing an additional material scaling parameter, materials which are very strongly anisotropic, such as unidirectional fiber reinforced composites which have very large variations in stiffness with direction, are not allowed. This does not eliminate all anisotropic materials. A good example of an anisotropic material for which the theory is entirely valid is paper, which has a modest variation of stiffness with direction (Johnson and Urbanik, 1984).

The functional in (1) is written using the second Piola–Kirchhoff stress and Green strain. It is possible to derive the same basic equations using the first Piola–Kirchhoff stress and the deformation gradient as conjugate variables. The strain scaling must then be introduced indirectly, but this presents no fundamental problem. In order to define force and moment resultants analogous to those used in shell theories, it is necessary to introduce second Piola–Kirchhoff stresses in the resulting equations. Since the scaled strains appear directly in the formulation based on (1), it seems more natural to use (1), rather than a functional using the deformation gradient as the strain measure.

In plate and shell theory it is common to invoke the Kirchhoff or Kirchhoff–Love hypothesis. This provides a set of constraints on the allowable kinematics. In the preceding development, there was no assumption made regarding any particular form the kinematics must take. Rather, constraint conditions were derived as part of the analysis. This is a new feature of this type of theory. Koiter (1960) begins with the assumption that the plate is in a state of plane stress, and uses constitutive equations to argue that the shear strains are negligible. This is also the approach taken by Berg and Johnson (1989). The theories presented here are plane stress, but only as a result of the assumption (17) on the functional form the loads may assume. Further, restrictions on the constitutive equations are derived as a result of strain and load scalings, and not assumed *a priori*. It is not necessary to assume the plate will have Kirchhoff-type kinematics, or to assume that the plate is in a state of plane stress. In essence, a complicated assumption on the plate's deformation has been replaced by some simple ones on the allowable strains and deflections. It is certainly preferable to make an assumption which can be verified directly, rather than one which can be verified only indirectly.

One final point which needs further discussion is the fact that the two nontrivial theories presented both include eqns (19), (20) and (21). These equations provide constraints between the lowest order deformations and the lowest order slopes. If a finite element solution of the nontrivial large deflection theories is to be attempted, the constraints will need to be satisfied either with Lagrange multipliers (stresses), or with penalties (stiffnesses). In either case, a single element will, in general, have to incorporate up to six nonlinear constraints (Berg, 1988).

The governing equations and boundary conditions in this investigation are based on uniform scalings in both in-plane directions. This gives rise to a theory which is valid in the interior of the plate. Near the edges of the plate, however, the need for a more general length scaling normal to the edge is anticipated. This more general asymptotic expansion of the governing equations gives rise to a boundary layer on the edges of the plate similar to that derived by Fung and Wittrick (1955). This boundary layer satisfies the function Ashwell and Reissner seems to have had in mind for their edge beams. Further discussion of the boundary layer will appear in a future paper.

REFERENCES

Ashwell, D. G. (1963). The boundary layer in the bending of inextensible plates and shells. *Q. J. Mech. Appl. Math.* **16**, 179.

Berg, L. J. (1988). Foundations of large deflection small strain plate theory including an inextensible theory. Ph.D. Dissertation, University of Wisconsin-Madison.

Berg, L. J. and Johnson, M. W. (1989). Large deflection small strain plate theory: application to the analysis of cellulosic structures. In *Mechanics of Cellulosic and Polymeric Materials* (Edited by R. W. Perkins), AMD-Vol. 99, p. 95. ASME, New York.

Coutris, N. and Monavon, A. (1988). Nonlinear plate theory: dimensional analysis and asymptotic formulation of the large scale problem. *Int. J. Engng Sci.* **26**(11), 1189.

Fung, Y. C. and Wittrick, W. H. (1955). A boundary layer phenomenon in the large deflexion of thin plates. *Q. J. Mech. Appl. Math.* **8**, 191.

Johnson, M. W. (1985). On the foundations of the nonlinear theory of the cylindrical deformation of thin elastic plates. *Int. J. Solids Structures* **21**(1), 11.

Johnson, M. W. and Urbanik, T. J. (1984). A nonlinear theory for elastic plates with application to characterizing paper properties. *J. Appl. Mech.* **51**, 146.

Koiter, W. T. (1960). A consistent first approximation in the general theory of thin elastic shells. In *Proceedings of the Symposium on the Theory of Thin Elastic Shells* (Edited by W. T. Koiter), p. 12. North-Holland, Delft.

Niordson, F. I. (1985). *Shell Theory*. North-Holland, Amsterdam.

Novozhilov, V. V. (1953). *Foundations of the Nonlinear Theory of Elasticity*. Graylock Press, Rochester, NY.

Oden, J. T. and Reddy, J. N. (1976). *Variational Methods in Theoretical Mechanics*. Springer-Verlag, Berlin.

Simmonds, J. G. (1979). Special cases of the nonlinear shell equations. In *Trends in Solid Mechanics* (Edited by J. F. Besseling and A. M. A. van der Heijden), p. 211. Delft University Press, Sijthoff and Noordhoff.

Simmonds, J. G. and Mann, J. E. (1986). *A First Look at Perturbation Theory*. Krieger, Malabar, FL.

Reissner, E. (1961). Note on finite inextensional deformation on shallow elastic shells. *J. Math. Phys.* **40**, 253.

Reissner, E. (1962). Variational considerations for elastic beams and shells. *ASCE J. Engng Mech.* **8**, 23.

van Dyke, M. (1975). *Perturbation Methods in Fluid Mechanics*. Parabolic Press, Stanford, CA.